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Convergence of Hermite and Hermite–Fejér Interpolation of Higher Order for Freud Weights

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We investigate weighted $L_p (0 < p < \infty)$ convergence of Hermite and Hermite–Fejér interpolation polynomials of higher order at the zeros of Freud orthogonal polynomials on the real line. Our results cover as special cases Lagrange, Hermite–Fejér and Krylov–Stayermann interpolation polynomials. © 2001 Academic Press

Key Words: Freud weight; Hermite–Fejér interpolation; Hermite interpolation; Krylov–Stayermann interpolation; L_p convergence; Lagrange interpolation.

1. INTRODUCTION AND STATEMENT OF RESULTS

We study mean convergence of *Hermite and Hermite–Fejér interpolatory polynomials of higher order* for Freud type weight functions on the real line. More precisely, let $X := \{x_{kn}\} \subset \mathbb{R}$,

$$-\infty < x_{nn} < x_{n-1,n} < \cdots < x_{2n} < x_{1n} < \infty, \qquad n = 1, 2, ...,$$

be a set of pairwise different nodes. Then for any real-valued function f on $\mathbb{R}$ and an integer $m \geqslant 1$, see ([25]), the Hermite–Fejér interpolation polynomial of higher order $H_{nm}(f, X)$ of degree $\leqslant nm-1$ with respect to X is defined by

$$\begin{cases} H_{nm}(f, X, x_{kn}) = f(x_{kn}), & 1 \leqslant k \leqslant n, \\ H_{nm}^{(t)}(f, X, x_{kn}) = 0, & 1 \leqslant t \leqslant m-1, 1 \leqslant k \leqslant n. \end{cases} \tag{1.1}$$

We note that by definition, H_{n1} are the Lagrange, H_{n2} the Hermite–Fejér and H_{n4} the Krylov–Stayermann interpolatory polynomials [7, 22, 23]. By (1.1), we may write for $x \in \mathbb{R}$,

$$H_{nm}(f, X, x) = \sum_{k=1}^{n} f(x_{kn})\, h_{knm}(X, x), \qquad n = 1, 2, \ldots .$$

The polynomials

$$h_k(X, x) := h_{knm}(X, x) = l_{kn}^m(X, x) \sum_{i=0}^{m-1} e_{iknm}(x - x_{kn})^i, \qquad 1 \leqslant k \leqslant n$$

are unique, of degree exactly $nm-1$ and satisfy the relations

$$h_k^{(t)}(X, x_{ln}) = \delta_{0t}\, \delta_{lk},\ 1 \leqslant k, l \leqslant n,\ 0 \leqslant t \leqslant m-1, \tag{1.2}$$

where for nonnegative integers u and v

$$\delta_{uv} := \begin{cases} 1, & u = v \\ 0, & u \neq v. \end{cases}$$

Here, $l_{kn}(X, x)$ are the well known fundamental Lagrange polynomials of degree $n-1$ given by

$$l_{kn}(X, x) := \frac{w_n(x)}{w_n'(x_{kn})(x - x_{kn})}, \qquad w_n(x) := \prod_{k=1}^{n} (x - x_{kn}).$$

If $f \in C^{(m-1)}(\mathbb{R})$, then the Hermite interpolation polynomial of higher order $\hat{H}_{nm}(f, X, x)$ of degree $\leqslant nm-1$ with respect to X is defined by

$$\hat{H}_{nm}^{(t)}(f, X, x_{kn}) := f^{(t)}(x_{kn}), \qquad 1 \leqslant k \leqslant n, \quad 0 \leqslant t \leqslant m-1.$$

We may write for $x \in \mathbb{R}$,

$$\hat{H}_{nm}(f, X, x) = \sum_{t=0}^{m-1} \sum_{k=1}^{n} f^{(t)}(x_{kn})\, h_{tk}(X, x), \qquad m = 1, 2, \ldots,$$

where

$$\begin{aligned} h_{tk}(X, x) &:= h_{tknm}(X, x) \\ &= l_{kn}^m(X, x) \frac{(x - x_{kn})^t}{t!} \sum_{i=0}^{m-1-t} e_{tiknm}(x - x_{kn})^i, \qquad 0 \leqslant t \leqslant m-1 \end{aligned}$$

is the unique polynomial of degree $nm-1$ satisfying

$$h_{tk}^{(i)}(X, x_{jn}) = \delta_{ti}\,\delta_{kj}, \qquad 0 \leqslant i, \quad t \leqslant m-1, \quad 1 \leqslant j, \quad k \leqslant n. \tag{1.3}$$

The coefficients $e_{ik} := e_{iknm}$ and $e_{tik} := e_{tiknm}$ may be obtained from the properties of h_k and h_{tk}, (1.2) and (1.3), see e.g. (2.6). It follows that we may write for any polynomial P of degree $\leqslant nm-1$, and $x \in \mathbb{R}$

$$P(x) = \hat{H}_{nm}(P, X, x) = H_{nm}(P, X, x) + \sum_{t=1}^{m-1} \sum_{k=1}^{n} P^{(t)}(x_{kn})\, h_{tk}(X, x). \tag{1.4}$$

In this paper, we are interested in investigating $L_p(0 < p < \infty)$ convergence of Hermite–Fejér and Hermite interpolation of higher order for an interpolatory matrix X whose lines are the zeros of a sequence of orthogonal polynomials with respect to a class of Freud weights on the real line. As special cases of our main results, we are able to recover known results on weighted Lagrange, Hermite and Hermite–Fejér interpolation for even Freud weights on the real line. In particular, we are also able to derive new results for Krylov–Stayermann interpolation and higher order processes for Freud weights on the real line for *arbitrary* fixed values of m. We thus believe that our main theorems provide a unified method by which all of the above results may be obtained.

More precisely, we are concerned with Freud weights w of the form $w = \exp(-Q)$ where:

- Q: $\mathbb{R} \to \mathbb{R}$ is even and continuous.
- $Q^{(2)}$ is continuous in $(0, \infty)$.
- $Q' \geqslant 0$ in $(0, \infty)$.
- There are constants A and B with $1 < A \leqslant B$ so that

$$A \leqslant \frac{d}{dx}(xQ'(x))/Q'(x) \leqslant B, \qquad x \in (0, \infty).$$

This class is large enough to cover the well known example

$$w_\beta(x) := \exp(-|x|^\beta), \qquad x \in \mathbb{R}, \quad \beta > 1$$

of which the Hermite weight w_2 is a special case.

For a given Freud weight w, we denote by

$$p_n(w^2, x) = \gamma_n(w^2)\, x^n + \cdots, \gamma_n(w^2) > 0, \qquad n \geqslant 0$$

the unique orthonormal polynomials satisfying

$$\int_{\mathbb{R}} p_n(w^2, x)\, p_m(w^2, x)\, w^2(x)\, dx = \delta_{mn}, \qquad m, n = 0, 1, 2, \ldots$$

and denote by

$$x_{n,n}(w^2) < x_{n-1,n}(w^2) < \cdots < x_{2,n}(w^2) < x_{1,n}(w^2)$$

their n real simple zeros. We henceforth set $X := \{x_{kn}(w^2)\}_{k=1}^{n} = \{x_{kn}\}_{k=1}^{n}$.

The subject of general orthogonal polynomials and weighted approximation on the real line and on finite intervals of the real line of positive length, is a rich and well established topic of research and we refer the reader to [3, 8, 15 17, 18] and the many references cited therein for a comprehensive account of this vast area and its applications.

The results in this paper are motivated, in part, by the following papers dealing with the theory of Lagrange, Hermite and Hermite–Fejér interpolation for weights on the real line and on finite intervals. In [11, 14, 16, 20] above authors studied weighted uniform and mean convergence of Lagrange interpolation for Freud weights on the real line while in [4, 10, 13, 20], mean convergence of Hermite–Fejér and Hermite interpolation processes for Freud weights on the real line were investigated. In [19, 23, 24, 26, 27], Sakai, Vértesi and Xu studied weighted uniform and mean convergence of Hermite and Hermite–Fejér interpolations of higher order at the zeros of Jacobi polynomials. Earlier work on Krylov–Stayermann interpolation for Jacobi polynomials can be found in [7, 22] and an interesting survey on this topic and related subjects may be found in [25]. Finally in [6], Kasuga and Sakai have recently investigated, in particular, convergence of Hermite–Fejér interpolation of higher order for the Freud weight of the form $w^2(x) = \exp(-x^m)$, $m = 2, 4, \ldots$.

Before stating our main results, we find it convenient to introduce some needed notation. First, we will henceforth suppress the dependence of the matrix X on the sequences of functions defined above. For example we will often write $H_{mn}(f, X, x) = H_{mn}[f](x)$ and adopt similar conventions for other sequences of functions. For any two sequences (b_n) and (c_n) of nonzero real numbers, we shall write

$$b_n \lesssim c_n,$$

if there exists a constant $C > 0$, independent of n such that

$$b_n \leqslant C c_n \qquad \text{for } n \text{ large enough}$$

and we shall write

$$b_n \sim c_n,$$

if $b_n \lesssim c_n$ and $c_n \lesssim b_n$. Similar notation will be used for functions and sequences of functions. Given $m \geqslant 1$ and $0 < p < \infty$, we will always set for every natural number n

$$(\log n)^*_{m,p} := \begin{cases} \log n, & mp \neq 4 \\ (\log n)^{1+1/p}, & mp = 4. \end{cases}$$

The symbol C will always denote an absolute positive constant which may take on different values at different times and Π_n will denote the class of polynomials of degree at most $n \geqslant 1$.

Finally, let $a_u(w^2) := a_u$, for $u > 0$, be the u-th Mhaskar–Rakhmanov–Saff number, which is the unique positive root of the equation

$$u = \frac{2}{\pi} \int_0^1 \frac{a_u t Q'(a_u t)}{\sqrt{1-t^2}}\, dt, \qquad u > 0.$$

Throughout, w will denote a Freud weight as defined above and a_u will denote the Mhaskar–Rakhmanov–Saff number for the weight w^2. Following are our main results.

THEOREM 1.1a. *Let* $0 < p < \infty$, $1 \leqslant m < 4$ *and let* $\Delta \in \mathbb{R}$, $\alpha > 0$ *and* $\hat{\alpha} := \min\{1, \alpha\}$. *Then the following hold:*

(A) *Suppose that for* $0 < p \leqslant 4/m$, *we have uniformly for* $n \geqslant C$

$$a_n^{-(\alpha+\Delta)+1/p} n^{m/6-1/3} \lesssim \frac{1}{(\log n)^*_{m,p}} \tag{1.5}$$

and

$$\hat{\alpha} + \Delta > \frac{1}{p}. \tag{1.6}$$

Then

$$\lim_{n \to \infty} \|(f(x) - H_{nm}[f](x))\, w^m(x)(1+|x|)^{-\Delta}\|_{L_p(\mathbb{R})} = 0 \tag{1.7}$$

for every continuous function $f\colon \mathbb{R} \to \mathbb{R}$ *satisfying*

$$\lim_{|x| \to \infty} |f(x)|\, w^m(x)(1+|x|)^{\alpha} = 0. \tag{1.8}$$

Moreover,

$$\lim_{n\to\infty} \|(f(x)-\hat{H}_{nm}[f](x))\, w^m(x)(1+|x|)^{-\Delta}\|_{L_p(\mathbb{R})}=0 \tag{1.9}$$

for every $f\in C^{(m-1)}(\mathbb{R})$ *satisfying* (1.8) *and*

$$\sup_{x\in\mathbb{R}} |f^{(t)}(x)\, w^m(x)(1+|x|)^{\alpha}|<\infty, \qquad t=1,2,\ldots,m-1. \tag{1.10}$$

(B) *Suppose that for* $p>4/m$*, we have uniformly for* $n\geqslant C$

$$a_n^{-(\alpha+\Delta)+1/p} n^{(m-1)/3-2/(3p)} \lesssim \left(\frac{1}{\log n}\right) \tag{1.11}$$

and

$$a_n^{-(\hat{\alpha}+\Delta)+1/p} n^{m/6-2/(3p)} \lesssim \left(\frac{1}{\log n}\right). \tag{1.12}$$

Then (1.7) *holds for continuous functions satisfying* (1.8) *and* (1.9) *holds for continuous functions satisfying* (1.8) *and* (1.10).

THEOREM 1.1b. *Let* $0<p<\infty$, $m\geqslant 4$ *and let* $\Delta\in\mathbb{R}$, $\alpha>0$ *and* $\hat{\alpha}:=\min\{1,\alpha\}$. *In addition, assume that uniformly for* $n\geqslant C$

$$a_n^{-\alpha} n^{m/6-1} \lesssim \frac{1}{(\log n)^{1/p}}. \tag{1.13}$$

Then the following hold:

(A) *Suppose that for* $0<p\leqslant 4/m$, (1.5) *and* (1.6) *hold. Then* (1.7) *holds for continuous functions satisfying* (1.8) *and* (1.9) *holds for continuous functions satisfying* (1.8) *and* (1.10).

(B) *Suppose that for* $4/m<p\leqslant 1$*, there exists* $\delta_1>0$ *and* $\delta_2>0$ *such that uniformly for* $n\geqslant C$

$$a_n^{-(\alpha+\Delta)+1/p} n^{(m-1)/3-2/3} \lesssim (n^{-\delta_1}) \tag{1.14}$$

and

$$a_n^{-(\hat{\alpha}+\Delta)+1/p} n^{m/6-2/3} \lesssim (n^{-\delta_2}). \tag{1.15}$$

Then (1.7) *holds for continuous functions satisfying* (1.8) *and* (1.9) *holds for continuous functions satisfying* (1.8) *and* (1.10).

(C) *Suppose that for* $p>1$, (1.11) *and* (1.12) *hold. Then* (1.7) *holds for continuous functions satisfying* (1.8) *and* (1.9) *holds for functions satisfying* (1.8) *and* (1.10).

Remark.

(a) It is instructive to briefly discuss the assumptions (1.5)–(1.6), (1.11)–(1.12) and (1.13)–(1.15). Firstly, it is well known, see ([18], Theorem 3.2.1), that for every polynomial $P_n \in \Pi_n$, $n \geqslant 1$ and for a given Freud weight w

$$\|P_n w\|_{L_\infty[-a_n, a_n]} = \|P_n w\|_{L_\infty(\mathbb{R})}.$$

Thus in particular for weighted approximation, it has become natural to impose minimal growth assumptions on the sequence a_n in order to establish convergence of interpolation operators in suitable weighted spaces on the real line, see [1, 2, 4, 11, 16, 20] and the references cited therein.

(b) For a Freud weights w, it is well known, see [8], that uniformly for $u \geqslant C$,

$$u^{1/B} \lesssim a_u \lesssim u^{1/A}$$

so that in particular, the assumption (1.13) only becomes significant for $m>6$. Indeed, it is easily seen that (1.13) is readily satisfied for $1 \leqslant m \leqslant 6$. If (1.6) holds, then $a_n^{-\alpha-\Delta+1/p}$ decreases to 0 for large n. If $p>4/m$, then it is easy to see that the exponents of n in (1.12) are positive. In particular, (1.12) implies (1.6). Similarly, if $m \geqslant 4$, (1.15) implies (1.6).

(c) In particular, for the weight $w=w_\beta$, it is well known, see [18, Chap. 4], that $a_n = Cn^{1/\beta}$ and thus we obtain the following result.

COROLLARY 1.2a. *Let* $w=w_\beta$, $\beta>1$, $0<p<\infty$ *and* $1 \leqslant m<4$. *In addition, let* $\Delta \in \mathbb{R}$, $\alpha>0$ *and* $\hat{\alpha} := \min\{1, \alpha\}$. *Then the following hold:*

(A) *Suppose that for* $0<p \leqslant 4/m$,

$$\frac{-\alpha}{\beta}+\frac{m}{6}<\frac{\Delta}{\beta}-\frac{1}{p\beta}+\frac{1}{3}; \qquad \hat{\alpha}+\Delta>\frac{1}{p}.$$

Then (1.7) *holds for continuous functions satisfying* (1.8) *and* (1.9) *holds for continuous functions satisfying* (1.8) *and* (1.10).

(B) *Suppose moreover that for* $p>4/m$ *we have*

$$\frac{-\alpha}{\beta}+\frac{m}{6}<\frac{\Delta}{\beta}-\frac{1}{p\beta}+\frac{2}{3p}-\frac{m}{6}+\frac{1}{3}; \qquad \frac{-\hat{\alpha}}{\beta}+\frac{m}{6}<\frac{\Delta}{\beta}-\frac{1}{p\beta}+\frac{2}{3p}.$$

Then (1.7) *holds for functions satisfying* (1.8) *and* (1.9) *holds for functions satisfying* (1.8) *and* (1.10).

COROLLARY 1.2b. *Assume the hypotheses of Corollary* 1.2a *except we assume that* $m \geqslant 4$. *Then the following hold:*

Suppose that for $0 < p \leqslant 4/m$, *we have*

$$\frac{-\alpha}{\beta}+\frac{m}{6}<\min\left\{1, \frac{\Delta}{\beta}-\frac{1}{p\beta}+\frac{1}{3}\right\}; \qquad \hat{\alpha}+\Delta>\frac{1}{p},$$

for $4/m < p \leqslant 1$, *we have*

$$\frac{-\alpha}{\beta}+\frac{m}{6}<\min\left\{1, \frac{\Delta}{\beta}-\frac{1}{p\beta}+\frac{2}{3}-\frac{m}{6}+\frac{1}{3}\right\}; \qquad \frac{-\hat{\alpha}}{\beta}+\frac{m}{6}<\frac{\Delta}{\beta}-\frac{1}{p\beta}+\frac{2}{3}$$

and for $p > 1$ *we have*

$$\frac{-\alpha}{\beta}+\frac{m}{6}<\min\left\{1, \frac{\Delta}{\beta}-\frac{1}{p\beta}+\frac{2}{3p}-\frac{m}{6}+\frac{1}{3}\right\}; \qquad \frac{-\hat{\alpha}}{\beta}+\frac{m}{6}<\frac{\Delta}{\beta}-\frac{1}{p\beta}+\frac{2}{3p}.$$

Then (1.7) *holds for continuous functions satisfying* (1.8) *and* (1.9) *holds for continuous functions satisfying* (1.8) *and* (1.10).

We observe that Theorems 1.1a and 1.1b allow us to recover as special cases, results on weighted Lagrange, Hermite, Hermite–Fejér and Krylov–Stayermann interpolation for Freud weights. For Lagrange, Hermite and Hermite–Fejér interpolation, special cases of our results for our class of weights have already appeared in [4, Theorem 1.1; 11, Theorem 1.3; 14, Theorem 1.1].

1.1. *Lagrange Interpolation: The Case* $m = 1$

COROLLARY 1.3. *Let* $0 < p < \infty$, $\Delta \in \mathbb{R}$, $a > 0$ *and* $\hat{\alpha} := \min\{1, \alpha\}$. *We assume that for* $0 < p \leqslant 4$,

$$\hat{\alpha}+\Delta>\frac{1}{p}$$

and for $p > 4$,

$$a_n^{-(\hat{\alpha}+\Delta)+1/p} n^{1/6(1-4/p)} \lesssim \left(\frac{1}{\log n}\right).$$

Then we have

$$\lim_{n\to\infty} \|(f(x)-L_n[f](x))\, w(x)(1+|x|)^{-\Delta}\|_{L_p(\mathbb{R})}=0$$

for every continuous function $f\colon \mathbb{R}\to\mathbb{R}$ *satisfying*

$$\lim_{|x|\to\infty} |f(x)|\, w(x)(1+|x|)^{\alpha}=0.$$

1.2. *Hermite and Hermite–Fejér Interpolation: The Case* $m=2$

COROLLARY 1.4. *Let* $0<p<\infty$, $\Delta\in\mathbb{R}$, $\alpha>0$ *and* $\hat{\alpha}:=\min\{1,\alpha\}$. *We assume that for* $0<p\leqslant 2$,

$$\hat{\alpha}+\Delta>\frac{1}{p}$$

and for $p>2$,

$$a_n^{-(\hat{\alpha}+\Delta)+1/p}n^{1/3(1-2/p)}\lesssim\left(\frac{1}{\log n}\right).$$

Then we have

$$\lim_{n\to\infty}\|(f(x)-H_{2n}[f](x))\,w^2(x)(1+|x|)^{-\Delta}\|_{L_p(\mathbb{R})}=0$$

for every continuous function $f\colon \mathbb{R}\to\mathbb{R}$ *satisfying*

$$\lim_{|x|\to\infty} |f(x)|\, w^2(x)(1+|x|)^{\alpha}=0. \tag{1.16}$$

Moreover,

$$\lim_{n\to\infty}\|(f(x)-\hat{H}_{2n}[f](x))\,w^2(x)(1+|x|)^{-\Delta}\|_{L_p(\mathbb{R})}=0$$

for every $f\in C^{(1)}(\mathbb{R})$ *satisfying* (1.16) *and*

$$\sup_{x\in\mathbb{R}} |f'(x)|\, w^2(x)(1+|x|)^{\alpha}<\infty.$$

1.3. *Krylov–Stayermann Interpolation: The Case* $m=4$

COROLLARY 1.5. *Let* $0<p<\infty$, $\Delta\in\mathbb{R}$, $\alpha>0$ *and* $\hat{\alpha}:=\min\{1,\alpha\}$. *We assume that for* $0<p\leqslant 1$,

$$a_n^{-(\alpha+\Delta)+1/p}n^{1/3}\lesssim\frac{1}{(\log n)^{*}_{4,p}}$$

and

$$\hat{\alpha}+\Delta>\frac{1}{p}.$$

Moreover for $p>1$ *assume*

$$a_n^{-(\alpha+\Delta)+1/p}n^{1-2/(3p)}\lesssim\left(\frac{1}{\log n}\right)$$

and

$$a_n^{-(\hat{\alpha}+\Delta)+1/p}n^{2/3-2/(3p)}\lesssim\left(\frac{1}{\log n}\right).$$

Then we have

$$\lim_{n\to\infty}\|(f(x)-K_{4n}[f](x))\,w^4(x)(1+|x|)^{-\Delta}\|_{L_p(\mathbb{R})}=0$$

for every continuous function $f\colon\mathbb{R}\to\mathbb{R}$ *satisfying*

$$\lim_{|x|\to\infty}|f(x)|\,w^4(x)(1+|x|)^{\alpha}=0. \tag{1.17}$$

Moreover,

$$\lim_{n\to\infty}\|(f(x)-\hat{K}_{4n}[f](x))\,w^4(x)(1+|x|)^{-\Delta}\|_{L_p(\mathbb{R})}=0$$

for every $f\in C^{(3)}(\mathbb{R})$ *satisfying* (1.17) *and*

$$\sup_{x\in\mathbb{R}}|f^{(t)}(x)|\,w^4(x)(1+|x|)^{\alpha}<\infty,\qquad t=1,2,3.$$

This paper is organized as follows. In Section 2, we state and prove a quadrature theorem which is of independent interest and in Section 3, we prove our main results. Section 4 contains an appendix with a technical lemma which we use throughout.

2. QUADRATURE AND DERIVATIVE ESTIMATES

In this section, we prove a quadrature estimate which is of independent interest. Throughout for convenience, we set for $n\geqslant 1$

$$x_{0,n}:=x_{1,n}+Cn^{-2/3}a_n,\qquad x_{n+1,n}:=x_{n,n}-Cn^{-2/3}a_n.$$

Following is our main result in this section:

THEOREM 2.1. *For* $\beta \in (0, 1/2)$, $\nu \in \mathbb{R}$, $r = 0, 1, 2, ..., m-1$ *and* $x \in \mathbb{R}$, *let*

$$\Sigma_r(x) := \left(\frac{n}{a_n}\right)^r \sum_{|x_{kn}| \geq \beta a_n} (|l_{kn}(x)|\, w^{-1}(x_{kn}))^m\, |x - x_{kn}|^r\, (1 + |x_{kn}|)^{-\nu}.$$

Then for some positive constants C_1, C_2 *and* C_3 *with* $x_{0,n} < (1 + C_2 n^{-2/3})\, a_n$, *we have uniformly for* $n \geq C$,

$$w^m(x)\, \Sigma_r(x) \lesssim a_n^{-\nu} \begin{cases} A_n(x), & |x| \leq \beta a_n/2 \\ B_n(x), & |x| \geq 2a_n \\ C_n(x), & \beta a_n/2 \leq |x| \leq a_n(1 - C_1 n^{-2/3}) \\ D_n(x), & a_n(1 - C_1 n^{-2/3}) \leq |x| \leq a_n(1 + C_2 n^{-2/3}) \\ E_n(x), & a_n(1 + C_2 n^{-2/3}) \leq |x| \leq 2a_n. \end{cases} \tag{2.1}$$

Here

$$A_n(x) := n^{\max\{m/6 - 1, 0\}} \begin{cases} \log n, & m = 6 \\ 1, & m \neq 6. \end{cases}$$

$$B_n(x) := a_n\, |x|^{-(m-r)}\, n^{\max\{m/6 - 1, 0\}} \begin{cases} \log n, & m = 6 \\ 1, & m \neq 6. \end{cases}$$

$$C_n(x) := (1 - |x|/a_n)^{-r/2} + n^{\max\{m/6 - 1/3, 0\}}\, |a_n^{1/2} p_n(x)\, w(x)|^m \log n.$$

$$D_n(x) := \left(\frac{n}{a_n}\right)^r ||x| - (1 - C_3 n^{-2/3})\, a_n|^r$$
$$+ n^{\max\{m/6 - 1/3, 0\}}\, |a_n^{1/2} p_n(x)\, w(x)|^m \log n.$$

$$E_n(x) := n^{\max\{m/6 - 1/3, 0\}}\, |a_n^{1/2} p_n(x)\, w(x)|^m \log n.$$

In order to prove Theorem 2.1, we need two auxiliary lemmas. We begin with:

LEMMA 2.2. *Let* n, $r > 1$. *Then uniformly for* $1 \leq k \leq n$,

$$\left|\frac{p_n^{(r)}(x_{kn})}{p_n'(x_{kn})}\right| \lesssim \left(\frac{n}{a_n}\right)^{r-1}. \tag{2.2}$$

For the weight $\exp(-x^m)$, m an even positive integer, Lemma 2.2 was first proved in [5, Lemma 4] for all $r \geq 1$. We emphasize that our method of proof differs from that used in [5] as there, heavy use was made of differential equations satisfied by the orthogonal polynomials in question.

Proof. We write

$$p_n(t) = l_{kn}(t)(t - x_{kn})\, p_n'(x_{kn}) \tag{2.3}$$

and introduce the reproducing kernel

$$K_n(x, t) := \sum_{k=0}^{n-1} p_k(x)\, p_k(t), \qquad x, t \in \mathbb{R}$$

and Cotes numbers

$$\lambda_{k,n} := K_n(x_{k,n}, x_{k,n})^{-1}, \qquad k \geqslant 1.$$

Then it is well known, see [3, Chap. 1], that for $1 \leqslant k \leqslant n$

$$K_n(t, x_{k,n}) = \frac{l_{k,n}(t)}{\lambda_{k,n}}, \qquad t \in \mathbb{R}$$

and for every polynomial P_{n-1} of degree at most $n-1$

$$P_{n-1}(x) = \int_{\mathbb{R}} P_{n-1}(t)\, K_n(t, x_{k,n})\, w^2(t)\, dt.$$

Applying these well known identities gives

$$\begin{aligned}
p_n^{(r)}(x_{kn}) &= \int_{\mathbb{R}} p_n^{(r)}(t)\, K_n(t, x_{kn})\, w^2(t)\, dt \\
&= \frac{1}{\lambda_{kn}} \int_{\mathbb{R}} p_n^{(r)}(t)\, l_{kn}(t)\, w^2(t)\, dt \\
&= \frac{p_n'(x_{kn})}{\lambda_{kn}} \int_{\mathbb{R}} (l_{kn}(t)(t - x_{kn}))^{(r)}\, l_{kn}(t)\, w^2(t)\, dt \\
&= \frac{p_n'(x_{kn})}{\lambda_{kn}} \int_{\mathbb{R}} (l_{kn}^{(r)}(t)(t - x_{kn})\, l_{kn}(t) + r l_{kn}^{(r-1)}(t)\, l_{kn}(t))\, w^2(t)\, dt \\
&= \frac{r p_n'(x_{kn})}{\lambda_{kn}} \int_{\mathbb{R}} l_{kn}^{(r-1)}(t)\, l_{kn}(t)\, w^2(t)\, dt.
\end{aligned}$$

Then by Hölder's inequality and Markov's inequality, see [9, Theorem 1,1] we learn that

$$\begin{aligned}
|p_n^{(r)}(x_{kn})| &\lesssim \frac{|p_n'(x_{kn})|}{\lambda_{kn}} \left(\int_{\mathbb{R}} (l_{kn}^{(r-1)}(t)\, w(t))^2\, dt \right)^{1/2} \left(\int_{\mathbb{R}} (l_{kn}(t)\, w(t))^2\, dt \right)^{1/2} \\
&= \frac{|p_n'(x_{kn})|}{\lambda_{kn}} \|l_{kn}^{(r-1)}(t)\, w(t)\|_{L_2(\mathbb{R})}\, \|l_{kn}(t)\, w(t)\|_{L_2(\mathbb{R})} \\
&\lesssim \frac{|p_n'(x_{kn})|}{\lambda_{kn}} \left(\frac{n}{a_n} \right)^{(r-1)} \|l_{kn}(t)\, w(t)\|_{L_2(\mathbb{R})}^2.
\end{aligned}$$

It remains to observe that

$$\frac{1}{\lambda_{k,n}} \|l_{kn}(t)\, w(t)\|_{L_2(\mathbb{R})}^2 = \int_{\mathbb{R}} K_n(t, x_{k,n})\, l_{k,n}(t)\, w^2(t)\, dt = l_{k,n}(x_{k,n}) = 1.$$

This completes the proof of (2.2). ∎

Next we use Lemma 2.2 to prove:

LEMMA 2.2. *Let* $r \geqslant 0$ *and* $n,\ m \geqslant 1$. *Then uniformly for* $1 \leqslant k \leqslant n$, $0 \leqslant t \leqslant m-1$ *and* $0 \leqslant s \leqslant m-1$

$$|[l_{kn}^m]^{(r)}(x_{kn})| \lesssim \left(\frac{n}{a_n} \right)^r \tag{2.4}$$

and

$$|e_{sk}| \lesssim \left(\frac{n}{a_n} \right)^s, \qquad |e_{tsk}| \lesssim \left(\frac{n}{a_n} \right)^s. \tag{2.5}$$

Proof. We prove (2.4) by induction on m. From (2.3) we easily obtain by using Leibnitz's rule for differentiation

$$l_{kn}^{(r)}(x_{kn}) = \frac{p_n^{(r+1)}(x_{kn})}{(r+1)\, p_n'(x_{kn})}$$

and so (2.4) holds for $m=1$ by Lemma 2.2. Now assume that (2.4) holds for $m=1, 2, \ldots, t-1$ for $t \geqslant 2$. Then using Leibnitz's rule for differentiation we obtain

$$\begin{aligned} |[l_{kn}^{t}]^{(r)}(x_{kn})| &\lesssim \sum_{i=0}^{r} \binom{r}{i} |l_{kn}^{(i)}(x_{kn})|\, |[l_{kn}^{t-1}]^{(r-i)}(x_{kn})| \\ &\lesssim \sum_{i=0}^{r} \binom{r}{i} \left(\frac{n}{a_n}\right)^{i} \left(\frac{n}{a_n}\right)^{(r-i)} \\ &\lesssim \left(\frac{n}{a_n}\right)^{r}. \end{aligned}$$

This completes the proof of (2.4). To prove (2.5), we proceed by induction on s. First for $s=0$, (2.5) is trivial since $e_{0k}=1$ and $e_{t0k}=1$. For $s \geqslant 1$, we have by (1.2)

$$0 = h_k^{(s)}(x_{kn}) = \sum_{i=0}^{s} e_{ik} \binom{s}{i} i! [l_{kn}^{m}]^{(s-i)}(x_{kn})$$

so that

$$e_{sk} = -\frac{1}{s!} \sum_{i=0}^{s-1} e_{ik} \binom{s}{i} i! [l_{kn}^{m}]^{(s-i)}(x_{kn}). \tag{2.6}$$

Thus if we assume that (2.5) holds for $s=0, 1, \ldots, t-1$ for $t \geqslant 1$, then by (2.6) and (2.4), we have

$$|e_{tk}| \lesssim \sum_{i=0}^{t-1} |e_{ik}|\, |[l_{kn}^{m}]^{(t-i)}(x_{kn})| \lesssim \sum_{i=0}^{t-1} \left(\frac{n}{a_n}\right)^{i} \left(\frac{n}{a_n}\right)^{t-i} \lesssim \left(\frac{n}{a_n}\right)^{t}.$$

By the same process for h_{tk}, we have $|e_{tsk}| \lesssim (\frac{n}{a_n})^s$. This completes the proof of Lemma 2.3. ∎

We now present the proof of Theorem 2.1:

Proof. For $|x_{kn}| \geqslant \beta a_n$, $|x_{kn}| \sim a_n$ by (4.2) so we may assume without loss of generality that $v=0$. We consider various cases:

Case 1. $|x| \leqslant \beta a_n/2$: First we observe that uniformly for $|x_{kn}| \geqslant \beta a_n$

$$|x - x_{kn}| \sim |x_{kn}| \sim a_n.$$

Moreover, for this range of x, (4.3) implies that

$$|a_n^{1/2} p_n(x)\, w(x)| \lesssim 1.$$

Thus (4.7) yields

$$
\begin{aligned}
&w^m(x)\,\Sigma_r\,(x)\\
&\quad\lesssim \left(\frac{n}{a_n}\right)^r\\
&\qquad\times \sum_{|x_{kn}|\geqslant \beta a_n} \left(\frac{a_n^{3/2}}{n}\max\{n^{-2/3},\,1-|x_{kn}|/a_n\}^{-1/4}\,\frac{|p_n(x)\,w(x)|}{|x-x_{kn}|}\right)^m |x-x_{kn}|^r\\
&\quad\lesssim \left(\frac{a_n}{n}\right)^{m-r} a_n^{r-m} \sum_{|x_{kn}|\geqslant \beta a_n} \max\{n^{-2/3},\,1-|x_{kn}|/a_n\}^{-m/4}.
\end{aligned}
\tag{2.7}
$$

Now using (4.2) we see that

$$
\begin{aligned}
&\sum_{|x_{kn}|\geqslant \beta a_n} \max\{n^{-2/3},\,1-|x_{kn}|/a_n\}^{-m/4}\\
&\qquad\lesssim \frac{a}{a_n}\sum_{|x_{kn}|\geqslant \beta a_n} \max\{n^{-2/3},\,1-|x_{kn}|/a_n\}^{-m/4+1/2}\,(x_{k-1,n}-x_{k+1,n})\\
&\qquad\lesssim \frac{n}{a_n}\,[\Sigma_{r^1}+\Sigma_{r^2}],
\end{aligned}
$$

where

$$
\Sigma_{r^1} := \sum_{\beta a_n\leqslant |x_{kn}|\leqslant (1-n^{-2/3})\,a_n} \max\{n^{-2/3},\,1-|x_{kn}|/a_n\}^{-m/4+1/2}\,(x_{k-1,n}-x_{k+1,n})
$$

and

$$
\Sigma_{r^2} := \sum_{|x_{kn}|\leqslant (1-n^{-2/3})\,a_n} \max\{n^{-2/3},\,1-|x_{kn}|/a_n\}^{-m/4+1/2}\,(x_{k-1,n}-x_{k+1,n}).
$$

Then we have by (4.1)

$$
\Sigma_{r^2} \lesssim n^{-2/3(-m/4+1/2)}\,|x_{0n}-(1-n^{-2/3})\,a_n| \lesssim a_n n^{m/6-1}
$$

and since $1-|x_{kn}|/a_n \sim 1-|t|/a_n$ for $t\in[x_{k+1,n},\,x_{k-1,n}]$ from (4.5), we have

$$
\begin{aligned}
\Sigma_{r^1} &\lesssim \sum_{\beta a_n\leqslant |x_{kn}|\leqslant (1-n^{-2/3})\,a_n} (1-|x_{kn}/a_n|)^{-m/4+1/2}\int_{x_{k+1,n}}^{x_{k-1,n}} dt\\
&\lesssim \int_{\beta a_n}^{(1-n^{-2/3})\,a_n} (1-|t|/a_n)^{-m/4+1/2}\,dt.
\end{aligned}
$$

Thus, we have

$$\sum_{|x_{kn}|\geqslant \beta a_n} \max\{n^{-2/3}, 1-|x_{kn}|/a_n\}^{-m/4}$$

$$\lesssim \frac{n}{a_n}\left[\int_{\beta a_n}^{(1-n^{-2/3})\,a_n} (1-t/a_n)^{-m/4+1/2}\,dt + a_n n^{m/6-1}\right]$$

$$\lesssim n^{1+\max\{m/6-1,\,0\}} \begin{cases} \log n, & m=6 \\ 1, & m\neq 6. \end{cases} \tag{2.8}$$

Substituting (2.8) into (2.7) proves Case 1.

Case 2. $|x|\geqslant 2a_n$: Here $|x-x_{kn}|\sim |x|$ and for this range of x,

$$|a_n^{1/2} p_n(x)\, w(x)| \lesssim 1$$

by (4.3). Thus using (2.8) and proceeding as in Case 1 gives

$$w^m(x)\,\Sigma_r(x)$$

$$\lesssim \left(\frac{n}{a_n}\right)^r$$

$$\times \sum_{|x_{kn}|\geqslant \beta a_n} \left(\frac{a_n^{3/2}}{n} \max\{n^{-2/3}, 1-|x_{kn}|/a_n\}^{-1/4} \frac{|p_n(x)\, w(x)|}{|x-x_{kn}|}\right)^m |x-x_{kn}|^r$$

$$\lesssim \left(\frac{a_n}{n}\right)^{m-r} |x|^{-(m-r)} \sum_{|x_{kn}|\geqslant \beta a_n} \max\{n^{-2/3}, 1-|x_{kn}|/a_n\}^{-m/4}$$

$$\lesssim a_n\, |x|^{-(m-r)}\, n^{\max\{m/6-1,\,0\}} \begin{cases} \log n, & m=6 \\ 1, & m\neq 6 \end{cases}$$

as required.

Case 3. $\beta a_n/2 \leqslant |x| \leqslant 2a_n$: We choose $l=l(x)$ such that $x\in[x_{l+1,n}, x_{ln}]$, if possible, and split

$$\Sigma_r(x) := \Sigma_{r^1}(x) + \Sigma_{r^2}(x),$$

where Σ_{r^1} sums over those k in Σ_r for which $k\in[l-3, l+3]$ and Σ_{r^2} contains the rest. Here, if $|x| > x_{0n}$, we set $\Sigma_{r^1} = 0$. Then we have much as in Cases 1 and 2

$$
\begin{aligned}
w^m(x)\, \Sigma_{r^2}\,(x) &\lesssim \left(\frac{n}{a_n}\right)^r \\
&\quad \times \Sigma_{r^2} \left(\frac{a_n^{3/2}}{n} \max\{n^{-2/3}, 1-|x_{kn}|/a_n\}^{-1/4} \frac{|p_n(x)\, w(x)|}{|x-x_{kn}|}\right)^m |x-x_{kn}|^r \\
&\lesssim \left(\frac{a_n}{n}\right)^{m-r-1} |a_n^{1/2} p_n(x)\, w(x)|^m\, \Sigma_{r^2} \frac{(x_{k-1,n}-x_{k+1,n})}{|x-x_{kn}|^{m-r}} \\
&\quad \times \max\{1-|x_{kn}|/a_n, n^{-2/3}\}^{-m/4+1/2}. \qquad (2.9)
\end{aligned}
$$

Then (2.9) becomes

$$
\begin{aligned}
w^m(x)\, \Sigma_{r^2}\,(x) &\lesssim \left(\frac{a_n}{n}\right)^{m-r-1} |a_n^{1/2} p_n(x)\, w(x)|^m\, n^{\max\{m/6-1/3,\,0\}}\, \Sigma_{r^2} \frac{(x_{k-1,n}-x_{k+1,n})}{|x-x_{kn}|^{m-r}} \\
&\lesssim \left(\frac{a_n}{n}\right)^{m-r-1} |a_n^{1/2} p_n(x)\, w(x)|^m\, n^{\max\{m/6-1/3,\,0\}} \\
&\quad \times \left[\int_{\beta a_n}^{x_{l+3,n}} + \int_{x_{l-3,n}}^{x_{0,n}} \frac{dt}{|x-t|^{m-r}} \right].
\end{aligned}
$$

Here, for $r < m-1$,

$$
\begin{aligned}
\int_{\beta a_n}^{x_{l+3,n}} + \int_{x_{l-3,n}}^{x_{0,n}} \frac{dt}{|x-t|^{m-r}} &\lesssim \int_{\beta a_n}^{x_{l+3,n}} \frac{dt}{(x-t)^{m-r}} + \int_{x_{l-3,n}}^{x_{o,n}} \frac{dt}{(t-x)^{m-r}} \\
&\lesssim (x_{l+1,n}-x_{l+3,n})^{-(m-r-1)} + (x_{l-3,n}-x_{l-1,n})^{-(m-r-1)} \\
&\lesssim \left(\frac{a_n}{n} \max\{1-|x|/a_n, n^{-2/3}\}^{-1/2}\right)^{-(m-r-1)} \\
&\lesssim \left(\frac{n}{a_n}\right)^{m-r-1} (n^{-2/3}+|1-|x|/a_n|)^{(m-r-1)/2} \\
&\lesssim \left(\frac{n}{a_n}\right)^{m-r-1}
\end{aligned}
$$

and for $r = m-1$

$$
\int_{\beta a_n}^{x_{l+3,n}} + \int_{x_{l-3,n}}^{x_{0,n}} \frac{dt}{|x-t|^{m-r}} \lesssim \log n.
$$

Therefore, for $r = 0, 1, 2, \ldots, m-1$

$$\int_{\beta a_n}^{x_{l+3,n}} + \int_{x_{l-3,n}}^{x_{0,n}} \frac{dt}{|x-t|^{m-r}} \lesssim \left(\frac{n}{a_n}\right)^{m-r-1} \log n.$$

Thus we have shown that for this range of x,

$$w^m(x)\, \Sigma_{r^2}(x) \lesssim n^{\max\{m/6-1/3,\, 0\}}\, |a_n^{1/2} p_n(x)\, w(x)|^m \log n.$$

Case 3.1. $\beta a_n/2 \leqslant |x| \leqslant (1-C_1 n^{-2/3})\, a_n$: We have

$$\begin{aligned} w^m(x)\, \Sigma_{r^1}(x) = \left(\frac{n}{a_n}\right)^r & ((l_{l+3,n}(x)\, w^{-1}(x_{l+3,n})\, w(x))^m\, |x-x_{l+3,n}|^r \\ & + \cdots + (l_{l-3,n}(x)\, w^{-1}(x_{l-3,n})\, w(x))^m\, |x-x_{l-3,n}|^r). \end{aligned}$$

Thus by (4.8) we have

$$w^m(x)\, \Sigma_{r^1}(x) \lesssim \left(\frac{n}{a_n}\right)^r |x_{l-3,n}-x_{l+3,n}|^r \sim (1-|x|/a_n)^{-r/2}.$$

Case 3.2. $(1-C_1 n^{-2/3})\, a_n \leqslant |x| \leqslant (1+C_2 n^{-2/3})\, a_n$: By a similar argument to the above we see that there exists a constant $C_3 > 0$ such that

$$w^m(x)\, \Sigma_{r^1}(x) \lesssim \left(\frac{n}{a_n}\right)^r ||x|-(1-C_3 n^{-2/3})\, a_n|^r.$$

Case 3.3. $(1+C_2 n^{-2/3})\, a_n \leqslant |x| \leqslant 2a_n$: Finally for this range of x, we observe that $\Sigma_{r^1}(x) = 0$. Combining all our estimates completes the proof of Theorem 2.1.

3. PROOF OF MAIN RESULTS

In this section we prove our main results, namely Theorems 1.1a and 1.1b. We find it convenient to split our functions to be approximated into pieces that vanish inside or outside $[-\beta a_n, \beta a_n]$ for some $\beta > 0$. For simplicity, we shall write

$$H_{n,m,i}[f](x) = H_{nmi}[f](x) := \sum_{k=1}^{n} e_{ik} l_{kn}^m(x)(x-x_{kn})^i f(x_{kn})$$

so that

$$H_{nm}[f](x) = \sum_{i=0}^{m-1} H_{nmi}[f](x).$$

We break up the proof of Theorems 1.1a and 1.1b into several lemmas. The first is given in:

LEMMA 3.1. *Let* $1 < p < \infty$, $\Delta \in \mathbb{R}$, $\alpha > 0$, $\hat{\alpha} := \min\{1, \alpha\}$ *and* $\varepsilon > 0$. *Let* $\beta \in (0, 1/2)$ *and assume further that* $\{f_n\}_{n=1}^{\infty}$ *is a sequence of measurable functions from* $\mathbb{R}$ *to* $\mathbb{R}$ *satisfying*

$$f_n(x) = 0, \qquad |x| < \beta a_n$$

and

$$|f_n w^m|(x) \leqslant \varepsilon(1+|x|)^{-\alpha}, \qquad x \in \mathbb{R} \quad \text{and} \quad n \geqslant 1. \tag{3.1}$$

Let $m \geqslant 1$.

(a) *Suppose for the given* m, $1 < p \leqslant 4/m$. *Then assume that* (1.5) *and* (1.6) *hold.*

(b) *Suppose that for the given* m, $p > 4/m$. *Then assume that* (1.11) *and* (1.12) *hold. Moreover, if* $m > 6$, *assume that* (1.13) *always holds.*

Then for $r = 0, 1, \ldots, m-1$, *we have*

$$\limsup_{n \to \infty} \|H_{nmr}[f_n](x)\, w^m(x)(1+|x|)^{-\Delta}\|_{L_p(\mathbb{R})} \lesssim \varepsilon.$$

Proof. First we have by (2.5), (3.1) and the definition of Σ_r in Theorem 2.1,

$$\begin{aligned}
&|H_{nmr}[f_n](x)\, w^m(x)(1+|x|)^{-\Delta}| \\
&\quad = \left| w^m(x) \sum_{k=1}^{n} e_{rk} l_{kn}^m(x)(x-x_{kn})^r f_n(x_{kn})(1+|x|)^{-\Delta} \right| \\
&\quad \lesssim \varepsilon w^m(x) \left(\frac{n}{a_n}\right)^r \sum_{|x_{kn}| \geqslant \beta a_n} |l_{kn}(x)\, w^{-1}(x_{kn})|^m \\
&\qquad \times |x-x_{kn}|^r (1+|x_{kn}|)^{-\alpha} (1+|x|)^{-\Delta} \\
&\quad = \varepsilon w^m(x)\, \Sigma_r(x)(1+|x|)^{-\Delta}.
\end{aligned} \tag{3.2}$$

Thus to prove Lemma 3.1 it suffices to estimate (3.2). We find it convenient to adopt the following notation. Set:

$$\begin{aligned}
A_1 &:= \{x \mid |x| \leqslant \beta a_n/2\}, \\
A_2 &:= \{x \mid |x| \geqslant 2a_n\}, \\
A_3 &:= \{x \mid \beta a_n/2 \leqslant |x| \leqslant (1-C_1 n^{-2/3})\, a_n\}, \\
A_4 &:= \{x \mid (1-C_1 n^{-2/3})\, a_n \leqslant |x| \leqslant (1+C_2 n^{-2/3})\, a_n\}, \\
A_5 &:= \{x \mid (1+C_2 n^{-2/3})\, a_n \leqslant |x| \leqslant 2a_n\}.
\end{aligned}$$

First by (3.2) and (2.1)

$$\begin{aligned}
\tau_n^{(n1)} &:= \|H_{nmr}[f_n](x)\, w^m(x)(1+|x|)^{-\Delta}\|_{L_p(A_1)} \\
&\lesssim \varepsilon a_n^{-\alpha} n^{\max\{m/6-1,\,0\}}\, \|(1+|x|)^{-\Delta}\|_{L_p(A_1)} \begin{cases} \log n, & m=6, \\ 1, & m\neq 6 \end{cases} \\
&\lesssim \varepsilon a_n^{-\alpha+\max\{-\Delta+1/p,\,0\}} n^{\max\{m/6-1,\,0\}} \begin{cases} \log n, & m=6, \\ 1, & m\neq 6. \end{cases} \\
&\quad \times \begin{cases} (\log n)^{1/p}, & \Delta p=1, \\ 1, & \Delta p\neq 1 \end{cases} \\
&\lesssim \varepsilon \begin{cases} a_n^{-\alpha}(\log n)^{1+1/p}, \quad (1) & m\leqslant 6,\ \Delta p\geqslant 1, \\ a_n^{-(\alpha+\Delta)+1/p}\log n, \quad (2) & m\leqslant 6,\ \Delta p<1, \\ a_n^{-\alpha} n^{m/6-1}(\log n)^{1/p}, \quad (3) & m>6,\ \Delta p\geqslant 1, \\ a_n^{-(\alpha+\Delta)+1/p} n^{m/6-1}, \quad (4) & m>6,\ \Delta p<1. \end{cases}
\end{aligned}$$

Case (a). Suppose $1<p\leqslant 4/m$ and (1.6) is satisfied. Then is suffices to consider the possibilities $m=1, 2, 3$.

If $\Delta p\geqslant 1$ then $(1)=O(1)$, since $\alpha>0$.

If $\Delta p<1$ then (1.6) implies $a_n^{-(\alpha+\Delta)+1/p}\leqslant a_n^{-(\hat{\alpha}+\Delta)+1/p}$, but here, $-(\hat{\alpha}+\Delta)+1/p<0$. Hence $(2)=O(1)$.

Case (b). If $p>4/m$,

if $m\leqslant 6$ and (1.12) is satisfied,

If $\Delta p\geqslant 1$, then $(1)=O(1)$, since $\alpha>0$;

if $\Delta p<1$, then (1.12) $\Rightarrow$ $(2)=O(1)$, because

$$\begin{aligned}
a_n^{-(\alpha+\Delta)+1/p}\log n &\leqslant a_n^{-(\hat{\alpha}+\Delta)+1/p}\log n \\
&\leqslant a_n^{-(\hat{\alpha}+\Delta)+1/p} n^{m/6p(p-4/m)}\log n \\
&= a_n^{-(\hat{\alpha}+\Delta)+1/p} n^{m/6-2/(3p)}\log n = O(1);
\end{aligned}$$

if $m>6$ and (1.12), (1.13) are satisfied,

if $\Delta p\geqslant 1$, then (1.13) $\Rightarrow$ $(3)=O(1)$;

if $\Delta p<1$, then (1.12) $\Rightarrow$ $(4)=O(1)$, because

$$\begin{aligned}
a_n^{-(\alpha+\Delta)+1/p} n^{m/6-1} &\leqslant a_n^{-(\hat{\alpha}+\Delta)+1/p} n^{m/6-1} \\
&= a_n^{-(\hat{\alpha}+\Delta)+1/p} n^{m/6-2/(3p)} n^{2/(3p)-1} \\
&= O\left(\frac{1}{\log n}\right) n^{-1+2/(3p)} = O(1).
\end{aligned}$$

Therefore, we have

$$\limsup_{n\to\infty} \tau_n^{(1)} \lesssim \varepsilon.$$

Next,

$$\begin{aligned}
\tau_n^{(2)} &:= \|H_{nmr}[f_n](x)\, w^m(x)(1+|x|)^{-\Delta}\|_{L_p(A_2)} \\
&\lesssim \varepsilon a_n^{-\alpha+1} n^{\max\{m/6-1,\,0\}} \begin{cases} \log n, & m=6, \\ 1, & m\neq 6. \end{cases} \\
&\quad \times \| |x|^{-(m-r)}(1+|x|)^{-\Delta}\|_{L_p(A_2)} \\
&\lesssim \varepsilon a_n^{-(\alpha+\Delta)+1/p-(m-r-1)} n^{\max\{m/6-1,\,0\}} \begin{cases} \log n, & m=6, \\ 1, & m\neq 6. \end{cases} \\
&\lesssim \varepsilon a_n^{-(\alpha+\Delta)+1/p} n^{\max\{m/6-1,\,0\}} \begin{cases} \log n, & m=6, \\ 1, & m\neq 6. \end{cases}
\end{aligned}$$

Case (a). If $1 < p \leqslant 4/m$ and (1.6) is satisfied.
Then $m < 6$ and (1.6) implies

$$a_n^{-(\alpha+\Delta)+1/p} n^{\max\{m/6-1,\,0\}} = a_n^{-(\alpha+\Delta)+1/p} \leqslant a_n^{-(\hat{\alpha}+\Delta)+1/p} = O(1).$$

Case (b). If $p > 4/m$ and (1.12) is satisfied,

if $m \leqslant 6$,

$$\begin{aligned}
&a_n^{-(\alpha+\Delta)+1/p} n^{\max\{m/6-1,\,0\}} \begin{cases} \log n, & m=6, \\ 1, & m\neq 6 \end{cases} \\
&\leqslant a_n^{-(\hat{\alpha}+\Delta)+1/p} \log n \\
&\leqslant a_n^{-(\hat{\alpha}+\Delta)+1/p} n^{m/6p(p-4/m)} \log n \\
&\leqslant a_n^{-(\hat{\alpha}+\Delta)+1/p} n^{m/6-2/(3p)} \log n = O(1);
\end{aligned}$$

if $m > 6$, (1.12) implies

$$\begin{aligned}
&a_n^{-(\alpha+\Delta)+1/p} n^{\max\{m/6-1,\,0\}} \begin{cases} \log n, & m=6, \\ 1, & m\neq 6 \end{cases} \\
&\leqslant a_n^{-(\hat{\alpha}+\Delta)+1/p} n^{m/6-1} \\
&\leqslant a_n^{-(\hat{\alpha}+\Delta)+1/p} n^{m/6-2/(3p)} n^{-1+2/(3p)} \\
&= O\left(\frac{1}{\log n}\right) n^{-1+2/(3p)} = O(1).
\end{aligned}$$

Therefore, we have

$$\limsup_{n\to\infty} \tau_n^{(2)} \lesssim \varepsilon.$$

Now we have

$$\begin{aligned}
\tau_n^{(3)} &:= \|H_{nmr}[f_n](x)\, w^m(x)(1+|x|)^{-\Delta}\|_{L_p(A_3)} \\
&\lesssim \varepsilon a_n^{-(\Delta+\alpha)} \left\| \left(1-\frac{|x|}{a_n}\right)^{-r/2} \right\|_{L_p(A_3)} \\
&\quad + \varepsilon a_n^{-(\Delta+\alpha)} n^{\max\{m/6-1/3,\,0\}} \log n\, \|(a_n^{1/2} p_n w)^m\|_{L_p(\mathbb{R})} \\
&\lesssim \varepsilon a_n^{-(\Delta+\alpha)} \left\| \left(1-\frac{|x|}{a_n}\right)^{-r/2} \right\|_{L_p(A_3)} \\
&\quad + \varepsilon a_n^{-(\Delta+\alpha)} n^{\max\{m/6-1/3,\,0\}} \log n\, \|a_n^{1/2} p_n w\|_{L_{mp}(\mathbb{R})}^m.
\end{aligned}$$

Observe that first

$$\begin{aligned}
\left\| \left(1-\frac{|x|}{a_n}\right)^{-r/2} \right\|_{L_p(A_3)} &\sim a_n^{1/p} \left(\int_{\beta/2}^{(1-C_1 n^{-2/3})} (1-t)^{-rp/2}\, dt \right)^{1/p} \\
&\sim a_n^{1/p} \begin{cases} 1, & rp<2, \\ (\log n)^{1/p}, & rp=2, \\ n^{-2/3(-r/2+1/p)}, & rp>2 \end{cases} \\
&\lesssim a_n^{1/p} n^{\max\{r/3-2/(3p),\,0\}} (\log n)^{1/p}
\end{aligned}$$

and second by (4.6)

$$\begin{aligned}
\|a_n^{1/2} p_n w\|_{L_{mp}(\mathbb{R})}^m &\lesssim a_n^{1/p} \begin{cases} 1, & mp<4, \\ (\log n)^{m/4}, & mp=4, \\ n^{m/6-2/(3p)}, & mp>4 \end{cases} \\
&\lesssim a_n^{1/p} n^{\max\{m/6-2/(3p),\,0\}} \begin{cases} (\log n)^{m/4}, & mp=4, \\ 1, & mp\neq 4. \end{cases}
\end{aligned}$$

Thus if $m \geqslant 2$, we have

$$\begin{aligned}
\tau_n^{(3)} &\lesssim \varepsilon a_n^{-(\Delta+\alpha)+1/p} (\log n)^{1/p}\, n^{\max\{r/3-2/(3p),\,0\}} \\
&\quad + \varepsilon a_n^{-(\Delta+\alpha)+1/p} n^{\max\{(m-1)/3-2/(3p),\, m/6-1/3\}} (\log n)_{m,p}^{*} \\
&= \varepsilon(b_n+c_n),
\end{aligned}$$

where

$$b_n := a_n^{-(\varDelta+\alpha)+1/p} (\log n)^{1/p}\, n^{\max\{r/3-2/(3p),\,0\}}$$

and

$$c_n := a_n^{-(\varDelta+\alpha)+1/p} n^{\max\{(m-1)/3-2/(3p),\, m/6-1/3\}} (\log n)^*_{m,\,p}.$$

Moreover if $m=1$ we have

$$\begin{aligned}\tau_n^{(3)} &\lesssim \varepsilon a_n^{-(\varDelta+\alpha)+1/p} \\ &\qquad + \varepsilon a_n^{-(\varDelta+\alpha)+1/p} n^{\max\{1/6-2/(3p),\,0\}} (\log n)^*_{1,\,p} \\ &= \varepsilon d_n,\end{aligned}$$

where

$$d_n := a_n^{-(\varDelta+\alpha)+1/p} + a_n^{-(\varDelta+\alpha)+1/p} n^{\max\{1/6-2/(3p),\,0\}} (\log n)^*_{1,\,p}.$$

First assume that $m \geqslant 2$. Then for b_n, we have

$$b_n \leqslant a_n^{-(\varDelta+\alpha)+1/p} (\log n)^{1/p}\, (1 + n^{(m-1)\,3-2/(3p)}).$$

Case (a). If $1 < p \leqslant 4/m$ and (1.5) are satisfied, then $(m-1)/3 - 2/(3p) \leqslant m/6 - 1/3$ and $2 \leqslant m < 4$, (1.5) implies

$$\begin{aligned}b_n &\lesssim a_n^{-(\varDelta+\alpha)+1/p} n^{m/6-1/3} (\log n)^{1/p} \\ &\leqslant a_n^{-(\varDelta+\alpha)+1/p} n^{m/6-1/3} (\log n)^*_{m,\,p} = O(1).\end{aligned}$$

Case (b). If $p > 4/m$ and (1.11), (1.12) are satisfied, then

$$\begin{aligned}b_n &\leqslant a_n^{-(\varDelta+\alpha)+1/p} (\log n)^{1/p}\, (1 + n^{(m-1)/3-2/(3p)}) \\ &\leqslant a_n^{-(\varDelta+\hat{\alpha})+1/p} n^{m/6p(p-4/m)} (\log n)^{1/p} \\ &\qquad + a_n^{-(\varDelta+\alpha)+1/p} n^{(m-1)/3-2/(3p)} (\log n)^{1/p} \\ &\leqslant a_n^{-(\varDelta+\hat{\alpha})+1/p} n^{m/6-2/(3p)} \log n \\ &\qquad + a_n^{-(\varDelta+\alpha)+1/p} n^{(m-1)/3-2/(3p)} \log n \\ &= O(1).\end{aligned}$$

For c_n,

Case (a). If $1 < p \leqslant 4/m$ and (1.5) are satisfied, then $(m-1)/3 - 2/(3p) \leqslant m/6 - 1/3$, (1.5) implies

$$c_n = a_n^{-(\varDelta+\alpha)+1/p} n^{m/6-1/3} (\log n)^*_{m,\,p} = O(1).$$

Case (b). If $p>4/m$ and (1.11) are satisfied, then $(m-1)/3-2/(3p)\geqslant m/6-1/3$, (1.11) implies

$$\begin{aligned} c_n &= a_n^{-(\Delta+\alpha)+1/p} n^{(m-1)/3-2/(3p)} (\log n)^*_{m,p} \\ &= a_n^{-(\Delta+\alpha)+1/p} n^{(m-1)/3-2/(3p)} \log n \\ &= O(1). \end{aligned}$$

Hence for $m\geqslant 2$, we have

$$\limsup_{n\to\infty} \tau_n^{(3)} \lesssim \varepsilon.$$

If $m=1$,

Case (a). If $1<p\leqslant 4$ and (1.6) is satisfied, then $1/6-2/(3p)\leqslant 0$ and (1.6) implies

$$\begin{aligned} d_n &= a_n^{-(\Delta+\alpha)+1/p} + a_n^{-(\Delta+\alpha)+1/p} n^{\max\{1/6-2/(3p),\,0\}} (\log n)^*_{1,p} \\ &\lesssim a_n^{-(\Delta+\alpha)+1/p} (\log n)^*_{1,p} \lesssim a_n^{-(\Delta+\hat{\alpha})+1/p} (\log n)^*_{1,p} = O(1). \end{aligned}$$

Case (b). If $p>4$ and (1.12) is satisfied, then $1/6-2/(3p)>0$ and (1.12) implies

$$\begin{aligned} d_n &\lesssim a_n^{-(\Delta+\hat{\alpha})+1/p} n^{\max\{1/6-2/(3p),\,0\}} (\log n)^*_{1,p} \\ &= a_n^{-(\Delta+\hat{\alpha})+1/p} n^{1/6-2/(3p)} \log n = O(1). \end{aligned}$$

Therefore, we have for $m=1$,

$$\limsup_{n\to\infty} \tau_n^{(3)} \lesssim \varepsilon.$$

We consider two further cases. First using Case 3

$$\begin{aligned} \tau_n^{(4)} &:= \| H_{nmr}[f_n](x)\, w^m(x)(1+|x|)^{-\Delta} \|_{L_p(A_4)} \\ &\lesssim \varepsilon a_n^{-(\Delta+\alpha)} \left(\frac{n}{a_n}\right)^r \| (|x|-(1-C_3 n^{-2/3})\, a_n)^r \|_{L_p(A_4)} \\ &\quad + \varepsilon \begin{cases} a_n^{-(\Delta+\alpha)+1/p} n^{\max\{(m-1)/3-2/(3p),\, m/6-1/3\}} (\log n)^*_{m,p}, & m\geqslant 2, \\ a_n^{-(\Delta+\alpha)+1/p} n^{\max\{1/6-2/(3p),\,0\}} (\log n)^*_{1,p}, & m=1. \end{cases} \end{aligned}$$

Since

$$\begin{aligned} &\| (|x|-(1-C_3 n^{-2/3})\, a_n)^r \|_{L_p(A_4)} \\ &\quad = \left(\int_{(1-C_1 n^{-2/3})\, a_n}^{(1+C_1 n^{-2/3})\, a_n} (|x|-(1-2C_3 n^{-2/3})\, a_n)^{rp}\, dx \right)^{1/p} \lesssim (n^{-2/3} a_n)^{r+1/p} \end{aligned}$$

it follows that we deduce

$$\tau_n^{(4)} \lesssim \varepsilon \begin{cases} a_n^{-(\Delta+\alpha)+1/p} n^{r/3-2/(3p)} \\ \qquad + a_n^{-(\Delta+\alpha)+1/p} n^{\max\{(m-1)/3-2/(3p),\, m/6-1/3\}} (\log n)^*_{m,p}, & m \geqslant 2, \\ a_n^{-(\Delta+\alpha)+1/p} + a_n^{-(\Delta+\alpha)+1/p} n^{\max\{1/6-2/(3p),\, 0\}} (\log n)^*_{1,p}, & m = 1, \end{cases}$$

$$\lesssim \varepsilon \begin{cases} b_n + c_n, & m \geqslant 2, \\ d_n, & m = 1. \end{cases}$$

Hence, much as in Case 3,

$$\limsup_{n\to\infty} \tau_n^{(4)} \lesssim \varepsilon.$$

Finally, we see that for $m \geqslant 2$, $\tau_n^{(5)} \lesssim \varepsilon c_n$ and for $m = 1$, $\tau_n^{(5)} \lesssim \varepsilon d_n$, where

$$\tau_n^{(5)} := \|H_{nmr}[f_n]\, w^m(x)(1+|x|)^{-\Delta}\|_{L_p(A_5)}.$$

Hence, we also have

$$\limsup_{n\to\infty} \tau_n^{(5)} \lesssim \varepsilon.$$

Therefore, we have for $r = 0, 1, ..., m-1$,

$$\limsup_{n\to\infty} \|H_{nmr}[f_n](x)\, w^m(x)(1+|x|)^{-\Delta}\|_{L_p(\mathbb{R})} \lesssim \varepsilon$$

and this last statement proves the lemma. ∎

Having dealt with functions that vanish inside $[-\beta a_n, \beta a_n]$, we turn to functions that vanish outside that interval.

We begin with:

LEMMA 3.2. *Let* $1 < p < \infty$, $\Delta \in \mathbb{R}$, $\alpha > 0$ *and* $\hat{\alpha} := \min\{1, \alpha\}$. *Let* $\beta \in (0, 1/2)$, $\varepsilon > 0$ *and assume that* $\{\psi_n\}_{n=1}^{\infty}$ *is a sequence of measurable functions from* $\mathbb{R}$ *to* $\mathbb{R}$ *satisfying*

$$\psi_n(x) = 0, \qquad |x| > \beta a_n$$

and

$$|\psi_n w^m|\,(x) \leqslant \varepsilon(1+|x|)^{-\alpha}, \qquad x \in \mathbb{R}, \quad n \geqslant 1. \tag{3.3}$$

Let $m \geqslant 1$.

(a) *Suppose for the given* m, $1<p\leqslant 4/m$. *Then assume that* (1.6) *holds.*

(b) *Suppose that for the given* m, $p>4/m$. *Then assume that* (1.12) *holds.*

Then for $r=O, 1, ..., m-1$,

$$\limsup_{n\to\infty} \|H_{nmr}[\psi_n](x)\, w^m(x)(1+|x|)^{-\Delta}\|_{L_p(|x|\geqslant 2\beta a_n)} \lesssim \varepsilon.$$

Proof. Indeed from (2.5), (3.3) and (4.7), we have for $|x|\geqslant 2\beta a_n$

$$\begin{aligned}
&|w^m(x)\, H_{nmr}[\psi_n](x)(1+|x|)^{-\Delta}|\\
&\lesssim a_n^{-\Delta} w^m(x) \left|\sum_{k=1}^{n} e_{rk} l_{kn}^m(x)\, \psi_n(x_{kn})(x-x_{kn})^r\right|\\
&\lesssim \varepsilon a_n^{-\Delta}\left(\frac{n}{a_n}\right)^r \sum_{|x_{kn}|\leqslant \beta a_n} |l_{kn}(x)\, w^{-1}(x_{kn})\, w(x)|^m\\
&\quad\times |x-x_{kn}|^r\, (1+|x_{kn}|)^{-\alpha}\\
&\lesssim \varepsilon a_n^{-\Delta}\left(\frac{n}{a_n}\right)^r\\
&\quad\times \sum_{|x_{kn}|\leqslant \beta a_n}\left(\frac{a_n^{3/2}}{n}\max\{n^{-2/3}, 1-|x_{kn}|/a_n\}^{-1/4}\frac{|p_n(x)\, w(x)|}{|x-x_{kn}|}\right)^m\\
&\quad\times |x-x_{kn}|^r\, (1+|x_{kn}|)^{-\alpha}\\
&\lesssim \varepsilon a_n^{-\Delta}\left(\frac{a_n}{n}\right)^{m-r} (a_n^{1/2} p_n(x)\, w(x))^m\\
&\quad\times \sum_{|x_{kn}|\leqslant \beta a_n} |x-x_{kn}|^{-(m-r)}\, (1+|x_{kn}|)^{-\alpha}\\
&\lesssim \varepsilon a_n^{-\Delta}\left(\frac{a_n}{n}\right)^{m-r-1} |x|^{-(m-r)}\, (a_n^{1/2} p_n(x)\, w(x))^m\\
&\quad\times \sum_{|x_{kn}|\leqslant \beta a_n} (1+|x_{kn}|)^{-\alpha}\, (x_{k-1,n}-x_{k+1,n})\\
&\lesssim \varepsilon a_n^{-\Delta}\left(\frac{a_n}{n}\right)^{m-r-1} |x|^{-(m-r)}\, (a_n^{1/2} p_n(x)\, w(x))^m\\
&\quad\times \int_{-2\beta a_n}^{2\beta a_n} (1+|t|)^{-\alpha}\, dt\\
&\lesssim \varepsilon a_n^{-\Delta}(a_n^{1/2} p_n(x)\, w(x))^m\, a_n^{-(m-r)} a_n^{1-\hat{\alpha}} \log n\\
&\lesssim \varepsilon a_n^{-(\hat{\alpha}+\Delta)}(a_n^{1/2} p_n(x)\, w(x))^m \log n.
\end{aligned}\tag{3.4}$$

If follows that using (3.4) and (4.6) we have

$$\|H_{nmr}[\psi_n](x)\, w^m(x)(1+|x|)^{-\Delta}\|_{L_p(|x|\geqslant 2\beta a_n)}$$
$$\lesssim \varepsilon a_n^{-(\hat{\alpha}+\Delta)+1/p} n^{\max\{m/6-2/(3p),\, 0\}}(\log n)^*_{m,p}.$$

Now observe that if $mp>4$,

$$\max\{m/6-2/(3p), 0\} = m/6-2/(3p).$$

Thus by (1.12), the polynomial growth of a_n and (1.6) we have,

$$\limsup_{n\to\infty} \|H_{nmr}[\psi_n](x)\, w^m(x)(1+|x|)^{-\Delta}\|_{L(|x|\geqslant 2\beta a_n)} \lesssim \varepsilon$$

and this proves the lemma. ▮

Next we present

Lemma 3.3. *Let $1<p<\infty$ and assume (1.6). Let $\varepsilon>0$, $\beta\in(0, 1/4)$ and assume that $\{\psi_n\}_{n=1}^{\infty}$ is a sequence of measurable functions from $\mathbb{R}$ to $\mathbb{R}$ satisfying*

$$\psi_n(x)=0, \qquad |x|>\beta a_n$$

and

$$|\psi_n w^m|\,(x) \leqslant \varepsilon(1+|x|)^{-\alpha}, \qquad x\in\mathbb{R}, \quad n\geqslant 1. \tag{3.5}$$

Then for $r=0, 1, ..., m-1$,

$$\limsup_{n\to\infty} \|H_{nmr}[\psi_n](x)\, w^m(x)(1+|x|)^{-\Delta}\|_{L_p(|x|\leqslant 2\beta a_n)} \lesssim \varepsilon.$$

Proof. We find it convenient to consider the estimation of the sequence of operators $H_{n,m,m-1}$ first and then the sequence $H_{n,m,r}$ for $r\leqslant m-2$. Thus let $|x|\leqslant 2\beta a_n$ and observe that using (4.3) we have

$$|a_n^{1/2} p_n(x)\, w(x)| \lesssim 1.$$

Thus for this range of $|x|$

$$|w^m(x)\, H_{n,m,m-1}[\psi_n](x)|$$
$$= \left|\sum_{k=1}^{n} e_{m-1,k} l_{kn}^m(x)\, w^m(x)(x-x_{kn})^{m-1}\, \psi_n(x_{kn})\right|$$
$$= \left|\sum_{k=1}^{n} e_{m-1,k} l_{kn}(x)\, w(x)(l_{kn}(x)\, w(x)(x-x_{kn}))^{m-1}\, \psi_n(x_{kn})\right|$$

$$= |p_n(x)\, w(x)|^{m-1} \left| \sum_{k=1}^{n} e_{m-1,k} l_{kn}(x)\, w(x)(p_n'(x_{kn}))^{(m-1)}\, \psi_n(x_{kn}) \right|$$

$$\lesssim \left| \sum_{k=1}^{n} e_{m-1,k} l_{kn}(x)\, w(x)(a_n^{1/2} p_n'(x_{kn}))^{-(m-1)}\, \psi_n(x_{kn}) \right|.$$

For each $n \geqslant 1$, we define two sequences of functions α_n and $\tilde{\psi}_n$ as follows: Set for $x \in \mathbb{R}$

$$\alpha_n(x) := \begin{cases} e_{m-1,k}(a_n^{1/2} p_n'(x_{kn}))^{-(m-1)}, & x = x_{k,n} k = 1, 2, ..., n \\ 0, & \text{otherwise} \end{cases}$$

and

$$\tilde{\psi}_n(x) := \psi_n(x)\, \alpha_n(x), \qquad x \in \mathbb{R} \quad \text{and} \quad n \geqslant 1.$$

Then clearly

$$\tilde{\psi}_n(x) = 0, \qquad |x| > \beta a_n. \tag{3.6}$$

Moreover, applying (2.5), (4.9) and (3.5) yields for $|x_{kn}| \leqslant \beta a_n$

$$|\tilde{\psi}_n(x_{kn})\, w(x_{kn})| \lesssim |\psi_n(x_{kn})|\, w^m(x_{kn}) \lesssim \varepsilon(1 + |x_{kn}|)^{-\alpha}. \tag{3.7}$$

Thus we have shown that for $|x| \leqslant 2\beta a_n$

$$|w^m(x)\, H_{n,m,m-1}[\psi_n](x)(1+|x|)^{-\Delta}| \lesssim \left| \sum_{k=1}^{n} l_{kn}(x)\, w(x)\, \tilde{\psi}_n(x_{kn})(1+|x|)^{-\Delta} \right|$$
$$= |L_n[\tilde{\psi}_n](x)\, w(x)(1+|x|)^{-\Delta}|,$$

where $\tilde{\psi}_n$ satisfy (3.6) and (3.7). Then applying ([11], Lemma 3.4) gives

$$\limsup_{n\to\infty} \|H_{n,m,m-1}[\psi_n](x)\, w^m(x)(1+|x|)^{-\Delta}\|_{L(|x| \leqslant 2\beta a_n)}$$
$$\lesssim \limsup_{n\to\infty} \|L_n[\tilde{\psi}_n](x)\, w(x)(1+|x|)^{-\Delta}\|_{L_p(|x| \leqslant 2\beta a_n)} \lesssim \varepsilon. \tag{3.8}$$

Next we turn to the estimation of the sequence of operators $H_{n,m,r}$ for $r \leqslant m-2$. Set

$$\hat{\psi}_n(x) := |\psi_n(x)|\, w^{m-2}(x), \qquad x \in \mathbb{R}, \quad n \geqslant 1.$$

Then it is easy to see that

$$\hat{\psi}_n(x) = 0, \qquad |x| > \beta a_n \tag{3.9}$$

and

$$|\hat{\psi}_n(x)\, w^2(x)| = |\psi_n(x)\, w^m(x)| \leqslant \varepsilon(1+|x|)^{-\alpha}, \qquad x \in \mathbb{R}. \tag{3.10}$$

Moreover for $r \leqslant m-2$ and $|x| \leqslant 2\beta a_n$, we apply (2.5) and obtain

$$\begin{aligned}
&|w^m(x)\, H_{nmr}[\psi_n](x)| \\
&\quad = \left| \sum_{k=1}^{n} e_{rk} l_{kn}^m(x)\, w^m(x)(x-x_{kn})^r\, \psi_n(x_{kn}) \right| \\
&\quad \lesssim \left(\frac{n}{a_n}\right)^r \sum_{k=1}^{n} |l_{kn}(x)\, w(x)(x-x_{kn})|^r\, |l_{kn}(x)\, w(x)|^{m-r-2}\, l_{kn}^2(x)\, w^2(x)| \\
&\qquad \times |\psi_n(x_{kn})|.
\end{aligned}$$

Since

$$|l_{kn}(x)\, w(x)(x-x_{kn})|^r = \left| \frac{p_n(x)\, w(x)}{p_n'(x_{kn})} \right|^r$$

and

$$|l_{kn}(x)\, w(x)|^{m-r-2} \lesssim w^{m-r-2}(x_{kn}),$$

we have

$$\begin{aligned}
&|w^m(x)\, H_{nmr}[\psi_n](x)| \\
&\quad \lesssim \left(\frac{n}{a_n}\right)^r \sum_{k=1}^{n} \left| \frac{p_n(x)\, w(x)}{p_n'(x_{kn})\, w(x_{kn})} \right|^r l_{kn}^2(x)\, w^2(x)\, w^{m-2}(x_{kn})\, |\psi_n(x_{kn})| \\
&\quad \lesssim \left(\frac{n}{a_n}\right)^r \sum_{k=1}^{n} \left(\frac{a_n}{n}\right)^r |a_n^{1/2} p_n(x)\, w(x)|^r\, l_{kn}^2(x)\, w^2(x)\, w^{m-2}(x_{kn})\, |\psi_n(x_{kn})| \\
&\quad \lesssim \sum_{k=1}^{n} l_{kn}^2(x)\, w^2(x)\, w^{m-2}(x_{kn})\, |\psi_n(x_{kn})| \\
&\quad = \sum_{k=1}^{n} l_{kn}^2(x)\, w^2(x)\, \hat{\psi}_n(x_{kn}).
\end{aligned}$$

Thus we have shown that

$$\begin{aligned}
&\|H_{nmr}[\psi_n](x)\, w^m(x)(1+|x|)^{-\Delta}\|_{L_p(|x| \leqslant 2\beta a_n)} \\
&\qquad \lesssim \left\| \sum_{k=1}^{n} l_{kn}^2(x)\, w^2(x)\, \hat{\psi}_n(x_{kn})(1+|x|)^{-\Delta} \right\|_{L_p(|x| \leqslant 2\beta a_n)},
\end{aligned}$$

where the sequence of functions $\hat{\psi}_n$ satisfy (3.9) and (3.10). Thus we may apply ([4], Lemma 3.3) and obtain for $r = 0, 1, ..., m-2$,

$$\limsup_{n \to \infty} \|H_{nmr}[\psi_n](x)\, w^m(x)(1+|x|)^{-\Delta}\|_{L_p(|x| \leqslant 2\beta a_n)} \lesssim \varepsilon. \tag{3.11}$$

Combining (3.8) and (3.11) proves Lemma 3.3. ∎

For $x \in \mathbb{R}$, let

$$\tilde{H}_{nmr}[f](x) := \left(\frac{n}{a_n}\right)^r \sum_{k=1}^{n} l_{kn}^m(x)(x - x_{kn})^r f(x_{kn}).$$

If we inspect the proofs of Lemma 3.1, Lemma 3.2, and Lemma 3.3, we see that they hold for this operator as well under all the hypotheses of these former lemmas and under the weaker condition that the real variable x in (3.1), (3.3) and (3.5) may be replaced by the subsequence $\{x_{kn}\}$, $k = 1, ..., n$. That is, for f,

$$|f(x_{kn})\, w^m(x_{kn})| \leqslant \varepsilon(1+|x_{kn}|)^{-\alpha}, \qquad k = 1, ..., n, \quad \alpha < 0.$$

With this observation, we prove our final lemma in this section, namely:

Lemma 3.4. *Let* $1 < p < \infty$, $\Delta \in \mathbb{R}$, $\alpha > 0$ *and* $\hat{\alpha} := \min\{1, \alpha\}$. *Let* $m \geqslant 1$ *and* $\varepsilon > 0$.

(a) *Suppose for the given m,* $1 < p \leqslant 4/m$. *Then assume that* (1.5) *and* (1.6) *hold.*

(b) *Suppose that for the given m,* $p > 4/m$. *Then assume that* (1.11) *and* (1.12) *hold. Moreover, if* $m > 6$, *assume that* (1.13) *always holds.*

Then for any fixed polynomial R,

$$\limsup_{|x| \to \infty} \|(H_{nm}[R](x) - R(x))\, w^m(x)(1+|x|)^{-\Delta}\|_{L_p(\mathbb{R})} \lesssim \varepsilon.$$

Proof. For any fixed polynomial R, by (4.4)

$$|R^{(t)}(x)\, w^m(x)(1+|x|)^{\alpha}| \leqslant M \qquad x \in \mathbb{R}, \quad t = 0, 1, ..., m-1.$$

where M is a constant independent of x and t. Then for $n \geqslant \deg R(x)$,

$$R(x) - H_{nm}[R](x) = \sum_{t=1}^{m-1} \sum_{k=1}^{n} R^{(t)}(x_{kn})\, h_{tk}(x).$$

Here, for $1 \leqslant t \leqslant m-1$

$$h_{tk}(x) = l_{kn}^m(x)\,\frac{(x-x_{kn})^t}{t!}\sum_{i=0}^{m-1-t} e_{tik}(x-x_{kn})^i$$

$$= \frac{1}{t!}\sum_{i=0}^{m-1-t} e_{tik}\, l_{kn}^m(x)(x-x_{kn})^{t+i}$$

$$= \frac{1}{t!}\sum_{i=0}^{m-1-t} \frac{e_{tik}}{\left(\frac{n}{a_n}\right)^{t+i}}\left(\frac{n}{a_n}\right)^{t+i} l_{kn}^m(x)(x-x_{kn})^{t+i}.$$

If we set

$$R_n^{[t,i]}(x) := R^{(t)}(x)\, r_n^{[t,i]}(x),$$

where $r_n^{[t,i]}(x)$ is a function satisfying

$$r_n^{[t,i]}(x_{kn}) = \frac{e_{tik}}{\left(\frac{n}{a_n}\right)^{t+i}} \qquad k = 1, 2, \ldots, n,$$

then for sufficiently large n,

$$|R_n^{[t,i]}(x_{kn})\, w^m(x_{kn})(1+|x_{kn}|)^\alpha|$$

$$= \left|\frac{e_{tik}}{\left(\frac{n}{a_n}\right)^{t+i}}\right| |R^{(t)}(x_{kn})\, w^m(x_{kn})(1+|x_{kn}|)^\alpha|$$

$$\lesssim \left|\frac{e_{tik}}{\left(\frac{n}{a_n}\right)^{t+i}}\right| \lesssim \left(\frac{n}{a_n}\right)^{-t} \leqslant \varepsilon.$$

Then

$$R(x) - H_{nm}[R](x)$$

$$= \sum_{t=1}^{m-1}\sum_{k=1}^{n} R^{(t)}(x_{kn})\,\frac{1}{t!}\sum_{i=0}^{m-1-t}\frac{e_{tik}}{\left(\frac{n}{a_n}\right)^{t+i}}\left(\frac{n}{a_n}\right)^{t+i} l_{kn}^m(x)(x-x_{kn})^{t+i}$$

$$= \sum_{t=1}^{m-1}\sum_{i=0}^{m-1-t}\frac{1}{t!}\sum_{k=1}^{n} R_n^{[t,i]}(x_{kn})\left(\frac{n}{a_n}\right)^{t+i} l_{kn}^m(x)(x-x_{kn})^{t+i}$$

$$= \sum_{t=1}^{m-1}\sum_{i=0}^{m-1-t}\frac{1}{t!}\tilde{H}_{n,m,t+i}[R_n^{[t,i]}](x)$$

and

$$\|(H_{nm}[R](x)-R(x))\, w^m(x)(1+|x|)^{-\Delta}\|_{L_p(\mathbb{R})}$$
$$\leqslant \sum_{t=1}^{m-1} \sum_{i=0}^{m-1-t} \frac{1}{t!} \|\tilde{H}_{n,m,t+i}[R_n^{[t,i]}](x)\, w^m(x)(1+|x|)^{-\Delta}\|_{L_p(\mathbb{R})}.$$

Let χ_n be the characteristic function of $[-a_n/4, a_n/4]$ and

$$R_n^{[t,i]} = \chi_n R_n^{[t,i]} + (1-\chi_n)\, R_n^{[t,i]} := f_n + \psi_n.$$

Then using the observation just before the statement of the lemma,

$$\limsup_{n\to\infty} \|(H_{nm}[R](x)-R(x))\, w^m(x)(1+|x|)^{-\Delta}\|_{L_p(\mathbb{R})} \lesssim \varepsilon. \quad \blacksquare$$

We are now ready to present the:

Proof of Theorems 1.1a *and* 1.1b. We assume firstly that $1<p<\infty$. Since the conditions of Theorem 1.1a and Theorem 1.1b ensure the assumptions of Lemma 3.1, Lemma 3.2 and Lemma 3.3, we will use the results of these lemmas in our proof. Given any $\varepsilon>0$, we may find a polynomial P satisfying

$$|f-P|(x)\, w^m(x)(1+|x|)^{\alpha} \leqslant \varepsilon, \qquad x\in\mathbb{R}.$$

Then for $n \geqslant C$, we may write

$$\|(f-H_{nm}[f])(x)\, w^m(x)(1+|x|)^{-\Delta}\|_{L_p(\mathbb{R})}$$
$$\leqslant \|(f-P)(x)\, w^m(x)(1+|x|)^{-\Delta}\|_{L_p(\mathbb{R})}$$
$$+\|(P-H_{nm}[P])(x)\, w^m(x)(1+|x|)^{-\Delta}\|_{L_p(\mathbb{R})}$$
$$+\|H_{nm}[P-f](x)\, w^m(x)(1+|x|)^{-\Delta}\|_{L_p(\mathbb{R})}.$$

Here, $(\alpha+\Delta)\, p \geqslant (\hat{\alpha}+\Delta)\, p>1$ so that first

$$\|(f-P)(x)\, w^m(x)(1+|x|)^{-\Delta}\|_{L_p(\mathbb{R})} \leqslant \varepsilon\, \|(1+|x|)^{-(\alpha+\Delta)}\|_{L_p(\mathbb{R})} \lesssim \varepsilon.$$

Moreover by Lemma 3.4, we have

$$\lim_{n\to\infty} \|(P-H_{nm}[P])(x)\, w^m(x)(1+|x|)^{-\Delta}\|_{L_p(\mathbb{R})} = 0.$$

Let χ_n be the characteristic function of $[-a_n/4, a_n/4]$ and let us write

$$P-f = (P-f)\, \chi_n + (P-f)(1-\chi_n) := \psi_n + f_n.$$

Then applying Lemmas 3.1–3.3 with $\beta = 1/4$ yields

$$\limsup_{n\to\infty} \|H_{nm}[P-f](x)\, w^m(x)(1+|x|)^{-\Delta}\|_{L_p(\mathbb{R})}$$

$$\leqslant \sum_{r=0}^{m-1} \limsup_{n\to\infty} \|H_{nmr}[P-f](x)\, w^m(x)(1+|x|)^{-\Delta}\|_{L_p(\mathbb{R})} \lesssim \varepsilon.$$

Thus

$$\limsup_{n\to\infty} \|(f-H_{nm}[f])(x)\, w^m(x)(1+|x|)^{-\Delta}\|_{L_p(\mathbb{R})} \lesssim \varepsilon$$

and so letting $\varepsilon \to 0+$ yields (1.7). To see (1.9), we apply the representation (1.4), the method of proof of Lemma 3.4 and (1.7). This completes the proof of Theorems 1.1a and 1.1b for the case $1 < p < \infty$.

Now, we assume that $0 < p \leqslant 1$.

The idea of the proof is simple. We first apply an idea of ([14], Theorem 1.1) whereby we reduce the problem to an application of Theorems 1.1a and 1.1b for $p > 1$. This is accomplished as follows. Let $q > 1$ and q' be its conjugate satisfying the relation

$$\frac{1}{q} + \frac{1}{q'} = 1.$$

Using Hölder's inequality, we observe that for any such q and any real Δ_1 we have the inequality

$$\|(f-H_{nm}[f](x))\, w^m(x)(1+|x|)^{-\Delta}\|^p_{L_p(\mathbb{R})}$$

$$= \int_{\mathbb{R}} |(f-H_{nm}[f](x))\, w^m(x)(1+|x|)^{-\Delta_1}\, (1+|x|)^{-(\Delta-\Delta_1)}|^p\, dx$$

$$\leqslant \left(\int_{\mathbb{R}} |(f-H_{nm}[f](x))\, w^m(x)(1+|x|)^{-\Delta_1}|^{pq}\, dx \right)^{1/q} \tag{3.12}$$

$$\times \left(\int_{\mathbb{R}} (1+|x|)^{-(\Delta-\Delta_1)\, pq'}\, dx \right)^{1/q'}. \tag{3.13}$$

Next we analyze the sufficient conditions (1.5)–(1.6), (1.11)–(1.12) and (1.14)–(1.15) carefully and prove the existence of a q with $pq > 1$ and Δ_1 so that Theorems 1.1a and 1.1b may be applied to (3.12). We will also show that with this careful choice of q and Δ_1, the term in (3.13) is also uniformly bounded. This will establish Theorems 1.1a and 1.1b for $0 < p < 1$ as required.

First, we consider the case $1 \leqslant m < 4$. Note that in this case we have $0 < p < 4/m$ and so we may choose q with $1 < pq < 4/m$. By (1.5) and (1.6), there exists some constant $A > 0$ such that for the given $n \geqslant C$

$$a_n^{-(\alpha+\Delta)+1/p} n^{m/6-1/3} (\log n)^*_{m,p} < A \tag{3.14}$$

and

$$a_n^{-(\hat{\alpha}+\Delta)+1/p} < 1. \tag{3.15}$$

From (3.14) and (3.15) we obtain respectively the relations

$$a_n^{-\alpha+1/pq} n^{m/6-1/3} (\log n)^*_{m,p} / A < a_n^{\Delta-1/p+1/pq}$$

and

$$a_n^{-\hat{\alpha}+1/pq} < a_n^{\Delta-1/p+1/pq}.$$

Thus from the above, we may choose Δ_1 satisfying

$$a_n^{-\alpha+1/pq} n^{m/6-1/3} (\log n)^*_{m,p} / A < a_n^{\Delta_1} < a_n^{\Delta-1/p+1/pq} \tag{3.16}$$

and

$$a_n^{-\hat{\alpha}+1/pq} < a_n^{\Delta_1} < a_n^{\Delta-1/p+1/pq}. \tag{3.17}$$

We summarize our findings as follows:
From the left most inequality in (3.16) we obtain the relation

$$a_n^{-(\alpha+\Delta_1)+1/pq} n^{m/6-1/3} (\log n)^*_{m,p} < A, \tag{3.18}$$

from the left most inequality in (3.17) we obtain the relation

$$-(\hat{\alpha}+\Delta_1)+1/pq < 0 \tag{3.19}$$

and finally from the right most inequality in (3.17) we obtain the relation

$$-(\Delta-\Delta_1)+1/p-1/pq < 0. \tag{3.20}$$

Thus (3.18) and (3.19) are just (1.5) and (1.6) respectively with p replaced by pq and Δ replaced by Δ_1. Thus Theorems 1.1a and 1.1b for the case $p > 1$ together with (3.20) ensure that Theorems 1.1a and 1.1b hold indeed for $0 < p < 1$ in this case.

Now, we consider the case $m \geqslant 4$. Clearly if $0 < p \leqslant 4/m$, we may apply exactly the same argument as above, so without loss of generality we assume that $4/m < p < 1$. We choose q with

$$1 < pq < \max\{1 - 3\delta_1/4, 1 - 3\delta_2/4, 0\}^{-1},$$

where δ_1 and δ_2 are as in (1.14) and (1.15). Then since

$$(\log n)^{-1/p} \leqslant (\log n)^{-1/pq},$$

we have

$$a_n^{-\alpha} n^{m/6-1} \lesssim (\log n)^{-1/pq}$$

and since (1.14) and (1.15) hold we also have the relations

$$a_n^{-(\alpha+\Delta)+1/p} n^{(m-1)/3-2/(3pq)} \log n < n^{2/3-2/(3pq)-\delta_1/2} < 1$$

and

$$a_n^{-(\hat{\alpha}+\Delta)+1/p} n^{m/6-2/(3pq)} \log n < n^{2/3-2/(3pq)-\delta_2/2} < 1.$$

From the above two relations we deduce that

$$a_n^{-\alpha+1/pq} n^{(m-1)/3-2/(3pq)} \log n < a_n^{\Delta-1/p+1/pq}$$

and

$$a_n^{-\hat{\alpha}+1/pq} n^{m/6-2/(3pq)} \log n < a_n^{\Delta-1/p+1/pq}.$$

Let us now choose Δ_1 satisfying

$$a_n^{-\alpha+1/pq} n^{(m-1)/3-2/(3pq)} \log n < a_n^{\Delta_1} < a_n^{\Delta-1/p+1/pq}$$

and

$$a_n^{-\hat{\alpha}+1/pq} n^{m/6-2/(3pq)} \log n < a_n^{\Delta_1} < a_n^{\Delta-1/p+1/pq}.$$

It follows that we have (1.11) and (1.12) with p replaced by pq and Δ replaced by Δ_1. Moreover (3.20) gain holds. Thus we conclude that

$$\lim_{n\to\infty} \int_{\mathbb{R}} |(f - H_{nm}[f](x))\, w^m(x)(1+|x|)^{-\Delta_1}|^{pq}\, dx = 0$$

and

$$\int_{\mathbb{R}} (1+|x|)^{-(\Delta-\Delta_1)\, pq'}\, dx < \infty.$$

Therefore,

$$\lim_{n\to\infty} \|(f - H_{nm}[f](x))\, w^m(x)(1+|x|)^{-\Delta}\|_{L_p(\mathbb{R})}^p = 0.$$

By the same method as above, we also have

$$\lim_{n\to\infty} \|(f-\hat{H}_{nm}[f](x))\, w^m(x)(1+|x|)^{-\Delta}\|^p_{L_p(\mathbb{R})} = 0.$$

This completes the proof of Theorems 1.1a and 1.1b. ∎

APPENDIX

In this last section we present a technical lemma concerning some estimates for the orthogonal polynomials for our class of weights. This lemma was use in Sections 2 and 3 and its statement in its present form can be found in [11, Theorems 2.1–2.2]. We emphasize that it is only included as a reference for easier reading.

LEMMA 4.1.

(a) *For* $n \geqslant 2$,

$$|1-x_{1n}/a_n| \lesssim n^{-2/3} \tag{4.1}$$

and uniformly for $1 \leqslant k \leqslant n-1$,

$$x_{k,n}-x_{k+1,n} \sim \frac{a_n}{n} \max\{1-|x_{k,n}|/a_n, n^{-2/3}\}^{-1/2}. \tag{4.2}$$

(b) *For* $n \geqslant 1$,

$$\sup_{x\in\mathbb{R}} |p_n(x)|\, w(x)\, |1-|x|/a_n|^{1/4} \sim a_n^{-1/2}. \tag{4.3}$$

and

$$\sup_{x\in\mathbb{R}} |p_n(x)|\, w(x) \sim n^{1/6} a_n^{-1/2}.$$

(c) *Let* $0 < p \leqslant \infty$. *For* $n \geqslant 1$ *and* $P \in \Pi_n$,

$$\|Pw\|_{L_p(\mathbb{R})} \lesssim \|Pw\|_{L_p[-a_n, a_n]}. \tag{4.4}$$

(d) *Uniformly for* $n \leqslant 2$ *and* $1 \leqslant k \leqslant n-1$,

$$(1-|x_{k,n}|/a_n) \sim (1-|x_{k+1,n}|/a_n). \tag{4.5}$$

(e) *Let* $0 < p < \infty$. *Uniformly for* $n \geqslant 1$,

$$\|p_n w\|_{L_p(\mathbb{R})} \sim a_n^{1/p-1/2} \times \begin{cases} 1, & p<4, \\ (\log n)^{1/4}, & p=4, \\ n^{(1/6)(1-4/p)}, & p>4. \end{cases} \tag{4.6}$$

(f) *Uniformly for* $n \geqslant 1$, $1 \leqslant k \leqslant n$, *and* $x \in \mathbb{R}$,

$$|l_{kn}(x)| \sim \frac{a_n^{3/2}}{n} w(x_{k,n}) \max\{n^{-2/3}, 1-|x_{k,n}|/a_n\}^{-1/4} \left|\frac{p_n(x)}{x-x_{k,n}}\right| \tag{4.7}$$

and

$$|l_{k,n}(x)|\, w^{-1}(x_{k,n})\, w(x) \lesssim 1. \tag{4.8}$$

(g) *Uniformly for* $n \geqslant 1$ *and* $1 \leqslant k \leqslant n$,

$$p_n'(x_{k,n})\, w(x_{k,n}) \sim \frac{n}{a_n^{3/2}} (\max\{n^{-2/3}, 1-|x_{k,n}|/a_n\})^{1/4}. \tag{4.9}$$

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REFERENCES

1. S. B. Damelin and D. S. Lubinsky, Necessary and sufficient conditions for mean convergence of Lagrange interpolation for Erdős weights, *Canad. J. Math.* **48** (1996), 710–736.
2. S. B. Damelin and D. S. Lubinsky, Necessary and sufficient conditions for mean convergence of Lagrange interpolation for Erdős weights II, *Canad. J. Math.* **48** (1996), 737–757.
3. G. Freud, "Orthogonal Polynomials," Pergamon Press, Oxford, 1971.
4. H. S. Jung and K. H. Kwon, Mean convergence of Hermite–Fejér and Hermits interpolation for Freud weights, *J. Comput. Appl. Math.* **99** (1998), 219–238.
5. Y. Kanjin and R. Sakai, Pointwise convergence of Hermite–Fejér interpolation of higher order for Freud weights, *Tohoku. Math.* **46** (1994), 181–206.
6. T. Kasuga and R. Sakai, Uniform or mean convergence of Hermite–Fejér interpolation of higher order for Freud weights, *J. Approx. Theory* **101** (1999), 330–358.
7. H. B. Knoop and B. Stockenberg, On Hermite–Fejér type interpolation, *Bull. Austral. Math. Soc.* **28** (1983), 39–51.
8. A. L. Levin and D. S. Lubinsky, Christoffel functions, orthogonal polynomials and Nevai's conjecture for Freud weights, *Constr. Approx.* **8** (1992), 461–533.

9. A. L. Levin and D. S. Lubinsky, L_p Markov–Bernstein inequalities for Freud weights, *J. Approx. Theory* **77** (1994), 229–248.
10. D. S. Lubinsky, Hermite and Hermite–Fejér interpolation and associated product integration rules on the real line: L_∞ theory, *J. Approx. Theory* **70** (1992), 284–334.
11. D. S. Lubinsky and D. M. Matjila, Necessary and sufficient conditions for mean convergence of Lagrange interpolation for Freud weights, *SIAM J. Math. Anal.* **26** (1995), 238–262.
12. D. S. Lubinsky and F. Móricz, The weighted L_p norms of orthogonal polynomials for Freud weights, *J. Approx. Theory* **77** (1994), 42–50.
13. D. S. Lubinsky and P. Rabinowitz, Hermite and Hermite–Fejér interpolation and associated product integration rules on the real line: L_1 theory, *Canad. J. Math.* **44** (1992), 561–590.
14. D. M. Matjila, Convergence of Lagrange interpolation for Freud weights in weighted $L_p(\mathbb{R})$, $0 < p < 1$, *in* "Nonlinear Numerical Methods and Rational Approximation II" (A. Cuyt, Ed.), pp. 25–35, Kluwer Academic, Dordrecht/Norwell, MA, 1994.
15. H. N. Mhaskar, "Introduction to the Theory of Weighted Polynomial Approximation," Series in Approximations and Decompositions, Vol. 7, World Scientific, Singapore.
16. P. Nevai, Mean convergence of Lagrange interpolation II, *J. Approx. Theory* **30** (1980), 263–276.
17. P. Nevai, Geza Freud, orthogonal polynomials and Christoffel functions: a case study, *J. Approx. Theory* **48** (1986), 3–167.
18. E. B. Saff and V. Totik, "Logarithmic Potentials with External Fields," Springer-Verlag, Heidelberg, 1997.
19. R. Sakai and P. Vértesi, Hermite–Fejér interpolations of higher order III, *Studia Sci. Math. Hungar.* **28** (1993), 87–97.
20. J. Szabados, Weighted Lagrange interpolation and Hermite–Fejér interpolation on the real line, *J. Inequality Appl.* **1** (1997), 99–123.
21. G. Szegő, "Orthogonal Polynomials," Amer. Math. Soc., New York, 1967.
22. P. Vértesi, Krylov–Stayerman polynomials on the Jacobi roots, *Studia Sci. Math. Hungar.* **22** (1987), 239–245.
23. P. Vértesi, Hermite–Fejér interpolations of higher order I, *Acta Math. Hungar.* **54** (1989), 135–152.
24. P. Vértesi, Hermite–Fejér interpolations of higher order II, *Acta Math. Hungar.* **56** (1990), 369–380.
25. P. Vértesi, Recent results on Hermite–Fejér interpolations of higher order (uniform metric), *Israel Math. Conf. Proc.* **4** (1991), 267–271.
26. P. Vértesi and Y. Xu, Weighted L_p convergence of Hermite interpolation of higher order, *Acta Math. Hungar.* **59** (1992), 423–438.
27. P. Vértesi and Y. Xu, Truncated Hermite interpolation polynomials, *Studia Sci. Math Hungar.* **28** (1993), 205–213.